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ON THE MOTION OF TWO SPHEROIDS IN AN INFINITE LIQUID ALONG THEIR COMMON AXIS OF REVOLUTION.

By Bibhutibhusan Datta.

Introduction.

Though the problem of the motion of two spheres in an infinite liquid along the line joining their centers has been completely solved by various investigators, the first writer to attempt the corresponding problem for two spheroids or ellipsoids is Prof. Karl Pearson.* His method does not, however, admit of further development and therefore, does not lead to the complete solution of the problem. In a previous paper,† I have shown how the problem can be completely solved in the case of two spheroids of revolution of *small* ellipticities, the motion of the solids being along their common axis of revolution. The present paper deals with the more *general* case of the same problem in as much as no limitation has been imposed as regards the ellipticities. It will be seen that the success of the problem depends on certain transformation theorems for spheroidal harmonics which were not known before, though the corresponding theorems for spherical harmonics were given long ago by Bessel.

All the results in this paper are believed to be new.

§ 1. Lemma.

1. Let r_1, θ_1, ω_1 and r_2, θ_2, ω_2 be two systems of spherical polar coördinates having their origins at O_1 and O_2 respectively and their polar axes in the directions of O_1O_2 and O_2O_1; also let x_1, y_1, z_1 and x_2, y_2, z_2 be two parallel systems of rectangular cartesian coördinates with the same origins, the polar axes being taken for the axes of x_1 and x_2; let s denote the distance between O_1 and O_2.

Since

$$\rho_2 = \sqrt{y_2^2 + z_2^2} = \rho_1$$

we have

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* "On the Motion of Spherical and Ellipsoidal Bodies in Fluid Media," Part II (*Quarterly Journal of Pure and Applied Mathematics*, Vol. XX).

† *Bulletin of the Calcutta Mathematical Society*, Vol. VII.

$$\frac{P_n(\cos\theta_2)}{r_2^{n+1}} = \frac{1}{\pi}\int_0^\pi \frac{d\vartheta}{(x_2+i\rho_2\cos\vartheta)^{n+1}}, \qquad i=\sqrt{-1}$$

$$= \frac{1}{\pi}\int_0^\pi \frac{d\vartheta}{\{s-(x_1+i\rho_1\cos\vartheta)\}^{n+1}};$$

when $|s| > |x_1 + i\rho_1\cos\vartheta|$, we can expand the right-hand side of this equation in a series of spherical harmonics,* so that

$$\frac{P_n(\cos\theta_2)}{r_2^{n+1}} = \frac{(-1)^n}{n!}\,\frac{1}{k_1^{n+1}}\cdot\frac{1}{\pi}\int_0^\pi\left[\sum_{m=0}^{\infty}(2m+1)P_m\left(\frac{x_1+i\rho_1\cos\vartheta}{k_1}\right)\frac{d^nQ_m(t)}{dt^n}\right]d\vartheta,$$

where $t = s/k_1$ and k_1 is any quantity.

Following the notation of Sir W. D. Niven in his memoir "On Ellipsoidal Harmonics" (*Phil. Trans.*, Vol. 128 (1891), *A*), let $\mathfrak{S}(2)$ denote the external harmonics of a system of confocal spheroids having the center at O_2 and O_1O_2 as the axis of revolution. Then $\mathfrak{S}(2)$ can be expressed in a series of spherical harmonics in the following way:†

$$\mathfrak{S}(2) = (-1)^n\frac{2^{n+1}n!}{(2n+1)!}H_n\left(\frac{\partial}{\partial x_2},\frac{\partial}{\partial y_2},\frac{\partial}{\partial z_2}\right)\left\{1+\frac{D^2}{2(2n+3)}\right.$$

$$\left.+\frac{D^4}{2\cdot4\cdot(2n+3)(2n+5)}+\cdots\right\}\frac{1}{\sqrt{x_2^2+y_2^2+z_2^2}},$$

where D^2 stands for $(a_2^2-c_2^2)(\partial^2/\partial x_2^2)$, and a_2, c_2 $(a_2 > c_2)$ are the semi-axes of any one of the quadrics of the second family.

Now

$$H_n\left(\frac{\partial}{\partial x_2},\frac{\partial}{\partial y_2},\frac{\partial}{\partial z_2}\right)\frac{1}{r_2} = (-1)^n\frac{(2n)!}{2^n n!}\,\frac{H_n(x_2,y_2,z_2)}{r_2^{2n+1}},\ddagger$$

and

$$H_n(x_2,y_2,z_2) = r_2^nP_n(\cos\theta_2).$$

Therefore

$$H_n\left(\frac{\partial}{\partial x_2},\frac{\partial}{\partial y_2},\frac{\partial}{\partial z_2}\right)\frac{1}{r_2} = (-1)^n\frac{(2n)!}{2^n n!}\,\frac{P_n(\cos\theta_2)}{r_2^n}$$

$$= (-1)^n\frac{(2n)!}{2^n n!}\frac{1}{\pi}\int_0^\pi\frac{d\vartheta}{(x_2+i\rho_2\cos\vartheta)^{n+1}}.$$

Hence if k_2^2 denotes $a_2^2-c_2^2$ we get

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* Vide Todhunter, "The Functions of Laplace, Lame and Bessel," p. 88.

† Vide Niven's memoir, p. 245; Hobson, "On the Evaluation of a Certain Surface Integral and its Application to the Expansion in Series of the Potential of Elipsoids," *Proc. Lond. Math. Soc.*, Vol. XXIV, p. 91.

‡ Niven's memoir, p. 236.

$$\mathfrak{S}(2) = \frac{2}{(2n+1)}\frac{1}{\pi}\int_0^\pi \Bigg[\frac{1}{(x_2+i\rho_2\cos\vartheta)^{n+1}} + \frac{(n+1)(n+2)}{2(2n+3)}\frac{k_2^2}{(x_2+i\rho_2\cos\vartheta)^{n+3}}+\cdots\Bigg]d\vartheta$$

$$= \frac{(2n)!}{2^{n-1}n!\,n!}\frac{1}{k_2^{n+1}}\frac{1}{\pi}\int_0^\pi Q_n\left(\frac{x_2+i\rho_2\cos\vartheta}{k_2}\right)d\vartheta.$$

Again

$$\mathfrak{S}(2) = \frac{2}{(2n+1)}\frac{(-1)^n}{n!}\frac{1}{k_1^{n+1}}\frac{1}{\pi}\int_0^\pi\sum_{m=0}^\infty(2m+1)P_m\left(\frac{x_1+i\rho_1\cos\vartheta}{k_1}\right)d\vartheta \times\left[\frac{d^n}{dt^n}+\frac{(k_2/k_1)^2}{2(2n+3)}\frac{d^{n+2}}{dt^{n+2}}+\cdots\right]Q_m(t).$$

Therefore we have

$$\frac{(2n+1)!}{2^n n!n!}\frac{1}{k_2^{n+1}}\frac{1}{\pi}\int_0^\pi Q_n\left(\frac{x_2+i\rho_2\cos\vartheta}{k_2}\right)d\vartheta$$

$$= \frac{(-1)^n}{n!}\frac{1}{k_1^{n+1}}\frac{1}{\pi}\int_0^\pi\sum_{m=0}^\infty(2m+1)P_m\left(\frac{x_1+i\rho_1\cos\vartheta}{k_1}\right)d\vartheta \times\left[\frac{d^n}{dt^n}+\frac{\sigma_1^2}{2(2n+3)}\frac{d^{n+2}}{dt^{n+2}}+\cdots\right]Q_m(t), \quad \text{(I)}$$

where σ_1 stands for k_2/k_1.

We shall now express this in terms of spheroidal coördinates. Supposing the spheroids are *prolate*, we write

$$x_1 = k_1\cos\theta_1\cosh\eta_1 = k_1\mu_1\lambda_1,$$
$$\rho_1 = k_1\sin\theta_1\sinh\eta_1 = k_1(1-\mu_1^2)^{1/2}(\lambda_1^2-1)^{1/2};$$

then

$$\frac{1}{\pi}\int_0^\pi P_m\left(\frac{x_1+i\rho_1\cos\vartheta}{k_1}\right)d\vartheta = P_m(\mu_1)P_m(\lambda_1),$$

and

$$\frac{1}{\pi}\int_0^\pi Q_n\left(\frac{x_2+i\rho_2\cos\vartheta}{k_2}\right)d\vartheta = P_n(\mu_2)Q_n(\lambda_2).$$

Substituting in (I) we get

$$P_n(\mu_2)Q_n(\lambda_2) = \sum_{m=0}^\infty(2m+1)\hat{\omega}_1(m,n)P_m(\mu_1)P_m(\lambda_1), \qquad (A)$$

where

$$\hat{\omega}_1(m,n) = (-1)^n\frac{2^n n!}{(2n+1)!}\sigma_1^{n+1}\left[\frac{d^n}{dt^n}+\frac{\sigma_1^2}{2(2n+3)}\frac{d^{n+2}}{dt^{n+2}}+\cdots\right]Q_m(t).$$

If however the spheroids are of the *oblate or planetary* form, $a_1^2 - c_1^2$ is negative, so that we write $\gamma_1^2 = c_1^2 - a_1^2$, and the corresponding coördinates are

$$x_1 = \gamma_1 \cos\theta_1 \sinh\eta_1 = \gamma_1\mu_1\lambda_1,$$

$$\rho_1 = \gamma_1 \sin\theta_1 \cosh\eta_1 = \gamma_1(1-\mu_1^2)^{1/2}(\lambda_1^2+1)^{1/2}.$$

Therefore (I) becomes

$$P_n(\mu_2)Q_n(i\lambda_2) = (-1)^{n+1}\sum_{m=0}^{\infty}(-1)^m(2m+1)\hat{\omega}_1'(m, n)P_m(\mu_2)P_m(i\lambda_2), \qquad (B)$$

where

$$\hat{\omega}_1'(m, n) = (-1)^n\frac{2^n n!}{(2n+1)!}\sigma_1'^{\,n+1}\left[\frac{d^n}{dt_1^n}+\frac{\sigma_1'^{\,2}}{2(2n+3)}\frac{d^{n+2}}{dt_1^{n+2}}+\cdots\right]Q_m(t_1)$$

$\sigma_1' = \gamma_2/\gamma_1$ and $t_1 = s/i\gamma_1$.

(A) and (B) are the lemmas that will be needed in the discussion of the problems that we shall presently take up.

2. We can put the expressions for $\hat{\omega}_1(m, n)$ and $\hat{\omega}_1(m, n)$ into more compact forms as definite integrals. For

$$\frac{d^nQ_m(t)}{dt^n} = \frac{(-1)^n n!}{2}\int_{-1}^{+1}\frac{P_m(p)dp}{(t-p)^{n+i}}.^*$$

Therefore

$$\hat{\omega}_1(m, n) = \frac{2^{n-1}n!\,n!}{(2n+1)!}$$

$$\times\int_{-1}^{+1}\left[\frac{\sigma_1^{n+1}}{(t-p)^{n+1}}+\frac{(n+1)(n+2)}{2(2n+3)}\frac{\sigma_1^{n+3}}{(t-p)^{n+3}}+\cdots\right]P_m(p)dp$$

$$= \frac{1}{2}\int_{-1}^{+1}Q_n\left(\frac{t-p}{\sigma_1}\right)P_m(p)dp.$$

Hence

$$\hat{\omega}_1(m, n) = \frac{1}{2}\int_{-1}^{+1}Q_n\left(\frac{s-x_1p}{k_2}\right)P_m(p)dp.$$

Similarly we obtain

$$\hat{\omega}_1'(m, n) = \frac{1}{2}\int_{-1}^{+1}Q_n\left(\frac{s-i\gamma_1p}{i\gamma_2}\right)P_m(p)dp.$$

§ 2. HYDRODYNAMICAL APPLICATION.

1. Let a_1, c_1 ($a_1 > c_1$) and a_2, c_2 ($a_2 > c_2$) be the semi-axes of two *ovary spheroids* having their centers at O_1 and O_2 respectively, O_1O_2 being the direction of motion.

Let u_1 be the velocity of O_1 towards O_2, u_2 that of O_2 towards O_1; also let s denote the distance between the centers O_1O_2 at any instant. Take λ_1, μ_1, ω and λ_2, μ_2, ω two systems of ovary spheroidal coördinates having their origins at O_1, O_2 respectively, so that the surfaces of the given spheroids are $\lambda_1 = \lambda_{10}$ and $\lambda_2 = \lambda_{20}$.

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* Todhunter, *loc. cit.*

The problem before us is to find a function ϕ satisfying the following conditions:

(1) $$\nabla^2\phi = 0, \text{ throughout the liquid,}$$

(2) $$\phi = 0, \text{ at infinity}$$

(3) $$\frac{\partial\phi}{\partial\lambda_1} = -\, u_1k_1P_1(\mu_1), \text{ when } \lambda_1 = \lambda_{10},$$

(4) $$\frac{\partial\,\phi}{\partial\lambda_2} = -\, u_2k_2P_1(\mu_2), \text{ when } \lambda_2 = \lambda_{20},$$

where $k_1^2 = a_1^2 - c_1^2$ and $k_2^2 = a_2^2 - c_2^2$.

A value of ϕ which satisfies the first two conditions is given by

(5) $$\phi = \sum_{n=1}^{\infty}[A_nP_n(\mu_1)Q_n(\lambda_1) + B_nP_n(\mu_2)Q_n(\lambda_2)].$$

If by a proper choice of the arbitrary constants A_n, B_n, we can make it satisfy the other two conditions, then ϕ will be the required solution.

By the Lemma (A), near the surface of O_1, we have

$$\phi = \sum_{n=1}^{\infty}\left[A_nP_n(\mu_1)Q_n(\lambda_1) + B_n\sum_{m=0}^{\infty}(2m+1)\hat{\omega}_1(m,\, n)P_m(\mu_1)P_m(\lambda_1)\right]$$

Substituting in the condition (3), we get

$$-\, u_1k_1P_1(\mu_1) = \sum_{n=1}^{\infty}\left[A_nP_n(\mu_1)Q_n'(\lambda_{10}) \right.$$
$$\left. + B\sum_{m=}^{\infty}(2m+1)\hat{\omega}_1(m,n)P_m(\mu_1)P_m'(\lambda_{10})\right],$$

where $Q_n'(\lambda_{10})$ and $P_n'(\lambda_{10})$ stand for the values of $(\partial/\partial\lambda_1)Q_n(\lambda_1)$ and $(\partial/\partial\lambda_1)P_n(\lambda_1)$ when λ_{10} has been substituted for λ_1. This must be true at every point on the surface of the spheroid. Hence we can equate to zero the coefficients of the various zonal harmonics of μ_1.

Equating the coefficients of $P_1(\mu_1)$

$$-\, u_1k_1 = A_1Q_1'(\lambda_{10}) + 3\sum_{n=1}^{\infty}\hat{\omega}_1(1,\, n)B_n. \tag{6}$$

Equating the coefficients of $P_p(\mu_1)$, $p > 1$,

$$O = A_pQ_p'(\lambda_{10}) + (2p+1)P_p'(\lambda_{10})\sum_{n-1}^{\infty}\hat{\omega}_1(p,\, n)B_n. \tag{7}$$

We shall have two similar equations from the remaining surface condition which can, however, be written down from symmetry. Thus

$$- u_2k_2 = B_1Q_1'(\lambda_{20}) + 3\sum_{n=1}^{\infty}\hat{\omega}_2(1, n)A_n, \tag{8}$$

for $p > 1$

$$0 = B_pQ_p'(\lambda_{20}) + (2p + 1)P_p'(\lambda_{20})\sum_{n=1}^{\infty}\hat{\omega}_2(p, n)A_n, \tag{9}$$

where $\hat{\omega}_2(m, n)$ can be written, by symmetry, from $\hat{\omega}_1(m, n)$, viz.

$$\hat{\omega}_2(m, n) = \frac{1}{2}\int_{-1}^{+1} Q_n\left(\frac{s - k_2p}{k_1}\right)P_m(p)dp.$$

Thus we have four equations (6) $\cdots$ (9) for the determination of the unknown constants A and B.

2. *The General Values of the Constants A and B.*—Substituting the values of B_1 and B_n $(n > 1)$ from the equations (8) and (9) in the equation (6), we get

$$- u_1k_1 = A_1Q_1'(\lambda_{10}) - 3\frac{\hat{\omega}_1(1, 1)}{Q_1'(\lambda_{20})}\left[u_2k_2 + 3\sum_{n=1}^{\infty}\hat{\omega}_2(1, n)A_n\right]$$
$$- 3\sum_{m=2}^{\infty}(2m + 1)\omega_1(1, m)\frac{P_m'(\lambda_{20})}{Q_m'(\lambda_{20})}\sum_{n=1}^{\infty}A_n\hat{\omega}_2(m, n),$$

or

$$\sum_{n=1}^{\infty}A_n\sum_{m=1}^{\infty}(2m + 1)\hat{\omega}_1(1, m)\hat{\omega}_2(m, n)\frac{P_m'(\lambda_{20})}{Q_m'(\lambda_{20})}$$
$$- \tfrac{1}{3}A_1Q_1(\lambda_{10}) = \tfrac{1}{3}u_1k_1 - u_2k_2\frac{\hat{\omega}_1(1, 1)}{Q_1(\lambda_{20})}.$$

This equation can be written in the form

$$\sum_{n=1}^{\infty}\theta_{nm}A_n - \tfrac{1}{3}A_1Q_1'(\lambda_{10}) = c, \qquad (m = 1, 2, 3, \cdots \text{ad inf}) \tag{1}$$

where

$$\theta_{nm} = (2m + 1)\hat{\omega}_1(1, m)\hat{\omega}_2(m, n)P_m'(\lambda_{20})/Q_m'(\lambda_{20}), \tag{2}$$

and

$$c = \tfrac{1}{3}u_1k_1 - u_2k_2\hat{\omega}_1(1, 1)/Q_1'(\lambda_{20}). \tag{3}$$

In a similar manner, for the determination of B's, we get the equations

$$\sum_{n=1}^{\infty}\theta_{nm}'B_n - \tfrac{1}{3}B_1Q_1'(\lambda_{20}) = c', \qquad (m = 1, 2, 3, \cdots \text{ad inf.}) \tag{4}$$

where

$$\theta_{nm}' = (2m + 1)\hat{\omega}_2(1, m)\hat{\omega}_1(m, n)P_m'(\lambda_{10})/Q_m'(\lambda_{10}), \tag{5}$$

and

$$c' = \tfrac{1}{3}u_2k_2 - u_1k_1\hat{\omega}_2(1, 1)/Q_1'(\lambda_{10}). \tag{6}$$

Thus we have finally an infinite number of equations (1) and (4) for the determination of an infinite number of unknowns, A as well as B.

The theory of the solution of the equations of this class has been worked out by Hill, Poincare, Koch, Toeplitz, Schmidt and others. Hence the values of the constants can be found so that the problems becomes determinate, the value of the potential function at any point being given by (5) Art. 1, § 2.

The general values of A as well as B, so determined will, however, be of little help for numerical calculations which will be necessary so as to get a physical idea of the state of liquid at any point. So we give a method of approximating to their values.

3. *Approximate Values of the Constants A and B.*—We have

$$\hat{\omega}_1(m,\, n) = (-\,1)^n \frac{2^n n\,!}{(2n+1)\,!}\sigma_1^{n+1}\left[\frac{d^n}{dt^n} + \frac{\sigma_1^2}{2(2n+3)}\frac{d^{n+2}}{dt^{n+2}} + \cdots\right] Q_m(t).$$

Substituting the value of $Q_m(t)$ in series, viz.

$$Q_m(t) = \frac{2^m m\,!\, m\,!}{(2m+1)\,!}\left[\frac{1}{t^{m+1}} + \frac{(m+1)(m+2)}{2(2m+3)}\frac{1}{t^{m+3}} + \cdots\right]$$

and rearranging the terms we get

$$\hat{\omega}_1(m,\, n) = \frac{2^{m+n} m\,!\, n\,!\,(m+n)\,!}{(2m+1)\,!\,(2n+1)\,!}\frac{k_1^m k_2^{n+1}}{s^{m+n+1}}$$
$$+ \frac{2^{m+n} m\,!\, n\,!\,(m+n+2)\,!}{(2m+1)\,!\,(2n+1)\,!\,2(2n+3)}\frac{k_1^m k_2^{n+1}}{s^{m+n+3}}(k_1^2 + 2k_2^2)$$
$$+ \cdots .$$

Thus the lowest order of $\hat{\omega}_1(m,\, n)$ is

$$\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^{m+n+1}$$

If the spheroids are so separated that we can neglect the terms of the order $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^3$, we get from the equation (1), Art. 1, § 2

$$A_1 = -\frac{u_1 k_1}{Q_1'(\lambda_{10})}; \quad \text{consequently} \quad B_1 = -\frac{u_2 k_2}{Q_2'(\lambda_{20})}, \quad \text{by symmetry.}$$

We see generally that, for $p > 1$

$$A_p = -\frac{2^{p+1} p\,!\,(p+1)\,!}{3\,!\,(2p)\,!}\frac{P_p'(\lambda_{10})}{Q_p'(\lambda_{10})}\frac{k_1^p k_2^2}{s^{p+2}}B_1 + \text{terms of higher orders,}$$

$$B_p = -\frac{2^{p+1} p\,!\,(p+1)\,!}{3\,!\,(2p)\,!}\frac{P_p'(\lambda_{20})}{Q_p'(\lambda_{20})}\frac{k_2^p k_1^2}{s^{p+2}}A_1 + \text{terms of higher orders.}$$

We proceed to approximate to the values of the A's and B's

First Approximation.—

$$A_2 = -\frac{2}{3}\frac{P_2'(\lambda_{10})}{Q_2'(\lambda_{10})}\frac{k_1^2k_2^2}{s^4}B_1, \qquad B_2 = -\frac{2}{3}\frac{P_2'(\lambda_{20})}{Q_2'(\lambda_{20})}\frac{k_2^2k_1^2}{s^4}A_1,$$

$$A_3 = -\frac{8}{15}\frac{P_3'(\lambda_{10})}{Q_3'(\lambda_{10})}\frac{k_1^3k_2^2}{s^5}B_1, \qquad B_3 = -\frac{8}{15}\frac{P_3'(\lambda_{20})}{Q_3'(\lambda_{20})}\frac{k_2^3k_1^2}{s^5}B_1,$$

. . . .

Second Approximation.—

$$A_1 = -\frac{u_1k_1}{Q_1'(\lambda_{10})} + \frac{2}{3}\frac{1}{Q_1'(\lambda_{10})Q_1'(\lambda_{20})}\frac{k_2k_2^3}{s^3}u_2,$$

$$B_1 = -\frac{u_2k_2}{Q_1'(\lambda_{20})} + \frac{2}{3}\frac{1}{Q_1'(\lambda_{20})Q_1'(\lambda_{10})}\frac{k_2k_1^3}{s^3}u_1,$$

. . . .

and so on.

4. *Verification of the Results Previously Obtained.*—Neglecting terms of the order $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^3$ and higher, we have found that

$$\phi = -\frac{u_1k_1}{Q_1'(\lambda_{10})}P_1(\mu_1)Q_1(\lambda_1) - \frac{u_2k_2}{Q_1'(\lambda_{20})}P_1(\mu_2)Q_1(\lambda_2). \tag{1}$$

Expressed in terms of spherical harmonics

$$P_1(\mu_1)Q_1(\lambda_1) = \frac{2}{3}k_1^2\left\{\frac{P_1(\cos\theta_1)}{r_1^2} + \frac{3}{5}k_1^2\frac{P_3(\cos\theta_1)}{r_1^4} + \cdots\right\},$$

$$P_1(\mu_2)Q_1(\lambda_2) = \frac{2}{3}k_2^2\left\{\frac{P_1(\cos\theta_2)}{r_2^2} + \frac{3}{4}k_2^2\frac{P_3(\cos\theta_2)}{r_2^4} + \cdots\right\};$$

also

$$Q_1'(x) = -\frac{2}{3}\left\{\frac{2}{x^3} + \frac{3}{5}\frac{4}{z^5} + \cdots\right\}.$$

Now if e_1 and e_2 be the eccentricities of the generating ellipses of the two spheroids respectively, we know

$$e_1 = 1/\lambda_{10}, \qquad e_2 = 1/\lambda_{20}, \qquad k_1 = a_1e_1, \qquad k_2 = a_2e_2.$$

Substituting in the equation (1) and neglecting powers of e_1 and e_2 higher than the second, we get as far as the required order

$$\phi = \tfrac{1}{2}u_1a_1^3(1 - \tfrac{6}{5}e_1^2)\frac{P_1(\cos\theta_1)}{r_1^2} + \tfrac{3}{10}u_1a_1^5e_1^2\frac{P_3(\cos\theta_1)}{r_1^4}$$

$$+ \tfrac{1}{2}u_2a_2^3\,(1 - \tfrac{6}{5}e_2^2)\frac{P_1(\cos\theta_2)}{r_2^2} + \tfrac{3}{10}u_2a_2^5e_2^2\frac{P_3(\cos\theta_2)}{r_2^4}.$$

Now from the results of the previous paper, we have neglecting powers of e_1 and e_2 higher than the second

$$A_1 = -\tfrac{1}{2}u_1 a^3\left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right), \qquad B_1 = -\tfrac{1}{2}u_2 b^3\left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right).$$

Making allowance for the difference of notations between the two papers we have

$$a = a_1(1-\tfrac{1}{3}e_1^2), \qquad b = a_2(1-\tfrac{1}{3}e_2^2), \qquad \epsilon_1 = \tfrac{1}{3}e_1^2, \qquad \epsilon_2 = \tfrac{1}{3}e_2^2.$$

Therefore

$$A_1 = -\tfrac{1}{2}u_1 a_1^3(1-\tfrac{6}{5}e_1^2), \qquad B_1 = -\tfrac{1}{2}u_2 a_2^3(1-\tfrac{6}{5}e_2^2).$$

Since the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^3$ are neglected,

$$A_2 = 0, \qquad A_3 = -\tfrac{1}{20}u_1 a_1^5 e_1^2, \qquad A_4 = 0, \qquad \cdots,$$

$$B_2 = 0, \qquad B_3 = -\tfrac{1}{20}u_2 a_2^5 e_2^2, \qquad B_4 = 0, \qquad \cdots.$$

Further

$$\frac{d^3}{ds_{12}^3}\left(\frac{1}{r_1}\right) = -3!\,\frac{P_3(\cos\theta_1)}{r_1^4}.$$

Hence the equation (v) becomes up to the order of terms retained

$$\phi = \tfrac{1}{2}u_1 a_1^3(1-\tfrac{6}{5}e_1^2)\frac{P_1(\cos\theta_1)}{r_1^2} + \tfrac{3}{10}u_1 a_1^5 e_1^2\frac{P_3(\cos\theta_1)}{r_1^4}$$
$$+ \tfrac{1}{2}u_2 a_2^3(1-\tfrac{6}{5}e_2^2)\frac{P_1(\cos\theta_2)}{r_2^2} + \tfrac{3}{10}u_2 a_2^5 e_2^2\frac{P_3(\cos\theta_2)}{r_2^4},$$

so that the results of the two papers agree at least up to this order. Similarly it can be shown that the two results agree in their most general form.

It may also be pointed out that, considered term for term, the series for the velocity potential function now obtained is more convergent than the series obtained in the previous paper.

5. *Conclusion.*—In the preceding articles, I have studied the case when the two spheroids have the same common axis of revolution which coincides also with the direction of motion. In a similar way, can be investigated the case when the axes of revolution of the two spheroids are at right angles to the direction of motion and the transformation theorem for spheroidal harmonics that will be required can be obtained.

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